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Necessary and Sufficient Stability Condition for Second-Order Switched Systems: A Phase Function Approach

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To find a unified approach for the stability analysis of second-order switched system, the concept of phase function is proposed in this paper. Firstly, the basic properties of phase function are explored. Following this concept and its properties, the phase-based stability criterion is investigated based on the Lyapunov theory, and a necessary and sufficient stability condition is obtained in the phase function approach. Moreover, the connection between phase-based stability conditions and algebraic condition of system matrices is also discussed. Finally, numerical examples are provided to exemplify the main result and make necessary comparisons with existing methods.

Keywords: Phase function, Switched system, Stability analysis, Lyapunov function

1. Introduction

A switched system is a kind of hybrid dynamical system consisting of a group of subsystems and a logic rule that arranges the switching among each subsystems (Lin & Antsaklis, 2009). This model is effective in describing the dynamics of numerous applications such as switching power converters (Serra, 2012), motor transmission systems (Naunheimer, Bertsche, Ryborz, & Novak, 2011) and supervisory control systems (Lin & Antsaklis, 2014). The detailed research motivations as well as some typical applications of switched systems can be found in the tutorial paper Lin & Antsaklis (2014).

The stability a switched system with arbitrary switching is equivalent to stability with time-varying uncertainty of such a system (Chesi, Garulli, Tesi, & Vicino, 2003). Among different approaches, the Lyapunov theory is proved to be an efficient tool for the stability analysis of switched systems. The level surfaces of a Lyapunov function are said to be the boundaries of positively invariant sets (Blanchini, 1999) of the systems. Application of positively invariant sets and its related principles are important in the construction of set-induced Lyapunov functions (Yfoulis & Shorten, 2004). Following this consideration, some novel techniques, such as norm-based Lyapunov function (Polanski, 1997), polyhedral Lyapunov function (Yfoulis & Shorten, 2004), line-integral Lyapunov function (Rhee & Won, 2006) and polynomial Lyapunov function (Tanaka, Yoshida, Ohtake, & Wang, 2009), have been proposed to reduce the conservativeness in stability analysis.

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When the investigated switched systems are restricted to the class of second-order, some special techniques can be applied. For example, analysis based on the polar coordinate transformation (see Godbehere & Sastry, 2010; Huang, Xiang, Lin, & Lee, 2010; Yang, Xiang, & Lee, 2012, 2014), the algebraic analysis for the existence of common quadratic Lyapunov functions (Shorten & Narendra, 2002), analysis by means of the generalized first integral (Margaliot & Langholz, 2003) and the geometry-based algorithm (Greco, Tocchini, & Innocenti, 2006). By these techniques, superior results can be obtained.

Generally, the Lyapunov methods (e.g. Tanaka et al., 2009; Xie, Shishkin, & Fu, 1997; Yfoulis & Shorten, 2004) provide a simple and unified approach for stability analysis. But the obtained results usually are just sufficient rather than necessary. Actually the conservativeness can be gradually reduced by choosing a more complex and flexible Lyapunov function, however, computational burden will be increased accordingly. Among these specific methods for the second-order case, the polar coordinate approach in Godbehere & Sastry (2010); Huang et al. (2010); Yang et al. (2012, 2014) and generalized first integral method in Margaliot & Langholz (2003) provide the possibility for necessary and sufficient condition. However, for both approaches, the discussion of subsystem matrices eigenvalue distribution cannot be avoided.

In this paper we focus on the second-order switched system and consider its necessary and sufficient stability condition under arbitrary control. To get rid of the discussion of eigenvalue types, the concept of phase function, which is a unified framework for stability analysis and control, is introduced in this paper and its properties are investigated accordingly.

Notations. The notations used throughout this paper are fairly standard. The superscript “ T ” stands for matrix transposition; \mathbb{R}^n denotes the n -dimensional Euclidean space; $\mathbb{R}_{>0}$ ($\mathbb{R}_{<0}$) denotes the set of positive (negative) real numbers; \mathbb{N}_0 denotes the set of non-negative integers; the notation $P > 0$ (≥ 0) means that P is real symmetric and positive definite (semi-definite); $a \stackrel{2\pi}{\equiv} b$ means $(a \bmod 2\pi) = (b \bmod 2\pi)$; $\|x\|_2$ is the 2nd norm of vector x ; $x \cdot y$ means the scalar product of vectors x and y . $\cot(\cdot)$ is the usual cotangent function, and $\csc(\cdot)$ is the usual cosecant function. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. Problem Formulation

Consider a second-order switched system

$$\dot{x}(t) = f_{\sigma(t)}(x(t)) = A_{\sigma(t)}x(t), \quad (1)$$

where $A_{\sigma(t)}$ can switch among a given collection of matrices A_1, A_2, \dots, A_q in $\mathbb{R}^{2 \times 2}$. Denote $\mathcal{Q} \triangleq \{1, 2, \dots, q\}$ as the set of indices of subsystems, then the switching signal in (1) can be constrained as $\sigma(t) \in \mathcal{Q}$. Throughout this paper, the input arguments of variables may be omitted to simplify the expressions, for example, $f_{\sigma(t)}(x(t))$ may be abbreviated as $f_{\sigma}(x)$ and $\theta(x)$ may be simplified as θ . For the above system, let us construct the following line-integral function (Rhee & Won, 2006),

$$V(x) \triangleq \int_{\Gamma(0,x)} p(\psi) \cdot d\psi, \quad (2)$$

where $\Gamma(0, x)$ is a path from the origin 0 to the current state x , $\psi \in \mathbb{R}^2$ is a dummy vector for the integral, $p(x) \in \mathbb{R}^2$ is a vector function of the state x , and $d\psi \in \mathbb{R}^2$ is an infinitesimal displacement vector. If $p(x)$ is regarded as a force vector at a state x , $V(x)$ in (2) can be regarded as the work done from the origin to the current state x , and is thus an energy-like function. To be a Lyapunov function candidate, $V(x)$ has to satisfy the following necessary conditions (Khalil, 2002),

- (a) $V(x)$ is continuously differentiable;
- (b) $V(x)$ is positive-definite;
- (c) $V(x)$ is radially unbounded.

Our purpose in this paper is to find a unified approach for stability analysis of second-order switched systems by applying the line-integral Lyapunov function. One important medium of this approach will be the new concept of *phase function*. The problems investigated in this paper can be stated as: Find the phase-based necessary and sufficient condition that guarantees the global stability of switched system (1) under arbitrary switching signals. Main contribution of this paper can be summarized as follows: (a) Analyzing the properties of phase function in the geometric way; (b) Describing the Lyapunov stability criteria in the form of phase function; (c) Obtaining the necessary and sufficient stability condition based on phase functions of subsystems.

Before discussing the stability analysis problem mentioned above, we will firstly introduce the concept of phase function, and investigate the intrinsic properties of it.

3. The Concept of Phase Function

3.1 Definition of phase function

The concept of phase function of second-order system is illustrated in the following definition. This concept will be used in the later sections as a new approach of stability analysis.

Definition 1: The *phase function* of a state-dependent non-zero vector $p(x) \in \mathbb{R}^2$ is defined as the angle from vectors x to $p(x)$, for all non-zero $x \in \mathbb{R}^2$. In the normal case this function is denoted as $\phi_p(x)$ with range $[0, 2\pi)$. In the symmetric case, it is denoted as $\phi_p^*(x)$ with range $[-\pi, \pi)$.

Firstly, we construct an angle function that is defined on the domain $\{x \mid 0 < x_1^2 + x_2^2\}$ and based on the the arctangent function $\text{atan2}(x_2, x_1)$ (Wikipedia, 2016) in computer science

$$\text{atan}(x) \triangleq \begin{cases} \text{atan2}(x_2, x_1), & x_2 \geq 0; \\ \text{atan2}(x_2, x_1) + 2\pi, & x_2 < 0; \end{cases}$$

According to Definition 1, the phase functions of vectors $f_\sigma(x)$ and $p(x)$ can be expressed as

$$\begin{aligned} \phi_\sigma(x) &\triangleq \text{atan}(f_\sigma(x)) - \text{atan}(x) \pmod{2\pi}, \\ \phi_p(x) &\triangleq \text{atan}(p(x)) - \text{atan}(x) \pmod{2\pi}, \end{aligned}$$

see Fig. 1. Note that, since $(t, x) \rightarrow f_{\sigma(t)}(x)$ is both state- and time-dependent, its phase function $\phi_\sigma(x)$ would also be time-dependent and is orchestrated by the switching signal $\sigma(t)$.

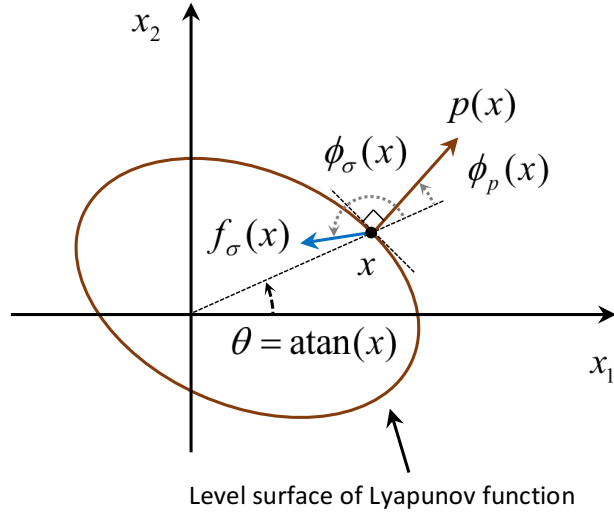


Figure 1.: Definition of phase functions $\phi_\sigma(x)$ and $\phi_p(x)$ at point x , with the oval curve being the level-surface of Lyapunov function $V(x)$ and $p(x)$ being the normal of level-surface at x

Specially for linear time-invariant system with system matrix A , we give the general expression of phase function with input matrix and angle variables A and θ ,

$$\varphi(A, \theta) \triangleq \text{atan}(A\omega(\theta)) - \theta \pmod{2\pi}. \quad (3)$$

The above definitions are dependent on the phase angle θ of system state x . In the subsequent analysis, we will also use their angle-dependent expressions $\varphi_p(\theta) \triangleq \phi_p(\omega(\theta))$, $\varphi_p^*(\theta) \triangleq \phi_p^*(\omega(\theta))$, $\varphi_\sigma(\theta) \triangleq \phi_\sigma(\omega(\theta))$ as the simplified versions of $\phi_p(\omega(\theta))$, $\phi_p^*(\omega(\theta))$, $\phi_\sigma(\omega(\theta))$, respectively, with

$$\omega(\theta) \triangleq [\cos \theta \quad \sin \theta]^T.$$

Generally, the inputs of functions defined with ϕ will be a vector, e.g., $\phi_\sigma(x)$, $\phi_p(x)$; and input of functions defined with φ will be an angle, e.g., $\varphi_\sigma(\theta)$, $\varphi_p(\theta)$.

3.2 Properties of phase function for linear systems

To apply phase function into stability analysis, we need to know the the properties of it. Firstly we want to check whether we can move the layout of phase function $\varphi(A, \theta)$ up, down, left and right by changing the parameters in matrix A . This shifting property can be analyzed based on the polar decomposition of matrix A . Clearly for any matrix A , we can always find its polar decomposition: right polar decomposition, $A = U_r P_r$, and left polar decomposition, $A = P_l U_l$, with U_r and U_l being unitary matrices, P_r and P_l being negative semidefinite symmetric matrices. What's more, for the obtained symmetric matrices P_r and P_l , we can make further decompositions $P_r = T_r^T \Lambda_r T_r$ and $P_l = T_l^T \Lambda_l T_l$, where Λ_r and Λ_l are diagonal matrices, and T_r and T_l are unitary matrices.

Overall we have

$$A = U_r P_r = U_r T_r^T A_r T_r, \quad A = P_l U_l = T_l^T A_l T_l U_l. \quad (4)$$

Remark 1: The unitary matrices would be easier to describe if they can be represented as rotation matrices. For the obtained unitary matrix U_r , one has $|\det(U_r)| = 1$. If $\det(U_r) = 1$, then U_r is called a *proper* unitary matrix (Wertz, 1978), which means U_r can be viewed as a rotation matrix. But if $\det(U_r) = -1$, then U_r would contain both rotation and reflection. On the other hand, matrix P_r is symmetric, so the unitary matrix T_r can always be intentionally constructed as a proper unitary matrix to express the effect of rotation, regardless of the eigenvalue distribution of P_r .

Note that $\det(P_r) \geq 0$ and $\det(P_l) \geq 0$ because P_r and P_l are chosen to be negative semidefinite symmetric. If A is Hurwitz, both U_r and U_l should be proper unitary matrices to ensure the relation that $\det(U_r) \det(P_r) = \det(P_l) \det(U_l) = \det(A) > 0$. Define a rotation matrix as $R(\theta) \triangleq \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$. Then in the case of Hurwitz A , the obtained unitary matrices in (4) can be expressed by rotation matrix $R(\theta)$ as

$$U_r \triangleq R(\alpha_r), \quad T_r \triangleq R(\beta_r), \quad U_l \triangleq R(\alpha_l), \quad T_l \triangleq R(\beta_l),$$

where $\alpha_r, \beta_r, \alpha_l, \beta_l \in [-\pi, \pi)$ are the corresponding rotation angles and the diagonal matrices in (4) can be denoted as

$$A_r \triangleq \text{diag}\{\lambda_{r1}, \lambda_{r2}\}, \quad A_l \triangleq \text{diag}\{\lambda_{l1}, \lambda_{l2}\},$$

where $\lambda_{r1}, \lambda_{r2}, \lambda_{l1}, \lambda_{l2} \in \mathbb{R}_{<0}$. Based on the decompositions in (4), one can find the following facts about the planar shifting property of phase function.

Lemma 3.1: *For any phase function $\varphi(A, \theta)$ with $\det(A) > 0$, the following results hold*

- (a) *Vertical shifting:* $\varphi(A, \theta) \stackrel{2\pi}{=} \varphi(P_r, \theta) + \alpha_r$;
- (b) *Diagonal shifting:* $\varphi(A, \theta) \stackrel{2\pi}{=} \varphi(P_l, \theta + \alpha_l) + \alpha_l$;
- (c) *Horizontal shifting:* $\varphi(A, \theta) = \varphi(U_r A_r, \theta + \beta_r) = \varphi(A_l U_l, \theta + \beta_l)$;

Proof. For (a), we can prove it by

$$\begin{aligned} \varphi(A, \theta) &\stackrel{2\pi}{=} \text{atan}(U_r P_r \omega(\theta)) - \text{atan}(\omega(\theta)) \\ &\stackrel{2\pi}{=} \text{atan}(U_r^T U_r P_r \omega(\theta)) - \text{atan}(U_r^T \omega(\theta)) \\ &\stackrel{2\pi}{=} \text{atan}(P_r \omega(\theta)) - \text{atan}(\omega(\theta - \alpha_r)) \\ &\stackrel{2\pi}{=} \varphi(P_r, \theta) + \alpha_r. \end{aligned}$$

Our next step is to prove (b). By simple derivation, one can find

$$\varphi(A, \theta) \stackrel{2\pi}{=} \text{atan}(P_l \omega(\theta + \alpha_l)) - \theta \stackrel{2\pi}{=} \varphi(P_l, \theta + \alpha_l) + \alpha_l,$$

thus (b) is proven. The proof of (c) will be a straightforward combination of (a) and (b). Note that unitary second-order matrices are commutative, which means

$$\varphi(A, \theta) = \varphi(U_r T_r^T \Lambda_r T_r, \theta) = \varphi(T_r^T U_r \Lambda_r T_r, \theta).$$

Combining the results in (a) and (b), one can get

$$\varphi(A, \theta) \stackrel{2\pi}{=} \varphi(U_r \Lambda_r T_r, \theta) - \beta_r \stackrel{2\pi}{=} \varphi(U_r \Lambda_r, \theta + \beta_r),$$

thus the result in (c) is proven. \square

Remark 2: All unitary matrices are commutative, so we can also write the polar decompositions of A as

$$A = T_r^T U_r \Lambda_r T_r \quad \text{and} \quad A = T_l^T \Lambda_l U_l T_l.$$

From this point of view, we may find that both T_r and T_l result in the horizontal shift of $\varphi(\cdot, \theta)$. But the rotation degrees β_r and β_l should be different since $U_r \Lambda_r \neq \Lambda_l U_l$. From the perspective of SVD decomposition $A = W \Sigma V^T$, we can assert that $U_r = U_l = V^T W$, $T_r = V^T$ and $T_l = W^T$. What's more, Λ_r and Λ_l should have the same eigenvalues, namely $\{\lambda_{r1}, \lambda_{r2}\} = \{\lambda_{l1}, \lambda_{l2}\}$, because both $\varphi(\Lambda_r, \theta)$ and $\varphi(\Lambda_l, \theta)$ have the same outline as $\varphi(A, \theta)$. Specially, if Λ_r and Λ_l are intentionally obtained as $\Lambda_r = \Lambda_l = \Lambda$, we can further get the relation that $U_l T_l = T_r$ or equivalently $\alpha_l + \beta_l = \beta_r$. The above relations are presented in Fig. 2. From Fig. 2, we can also confirm that, in vertical direction, rotation matrix with positive angle leads to upper shift. In the horizontal direction, rotation matrix with positive angle gives rise to the left shift.

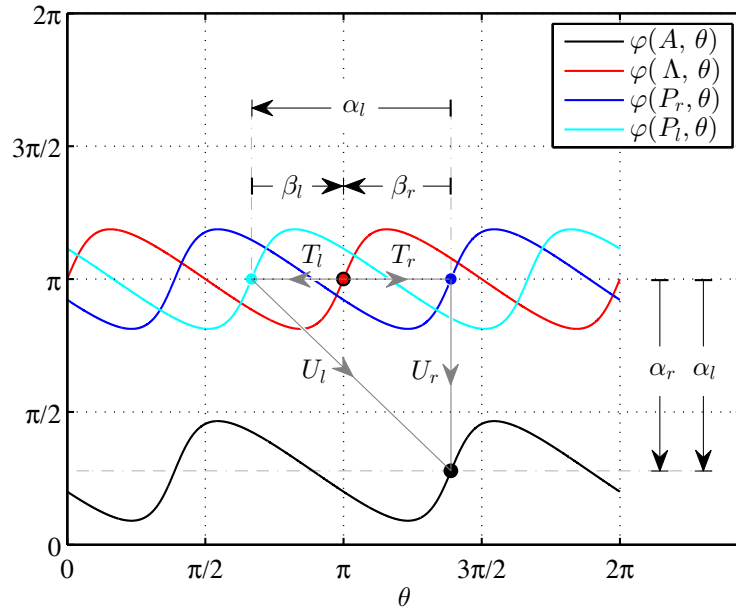


Figure 2.: Relation of right and left polar decompositions

From Lemma 3.1 and its following remarks, we can get a general impression of the layout and position of phase function $\varphi(A, \theta)$, as well as its relation with the polar decomposition of A . Overall, the outline of $\varphi(A, \theta)$ can be uniquely determined by parameters α_r, β_r and the ratio $\lambda_{r2}/\lambda_{r1}$. The properties introduced in the following lemmas will explain the relation between phase function $\varphi(A, \theta)$ and eigenvalues of matrix A .

Lemma 3.2: *For any $A \in \mathbb{R}^{2 \times 2}$ with $\det(A) > 0$, the following results hold*

- (a) *Positive real eigenvalue of A : $\varphi(A, \theta^*) = 0 \Leftrightarrow \exists \lambda \in \mathbb{R}_{>0}$ s.t. $A\omega(\theta^*) = \lambda\omega(\theta^*)$;*
- (b) *Negative real eigenvalue of A : $\varphi(A, \theta^*) = \pi \Leftrightarrow \exists \lambda \in \mathbb{R}_{>0}$ s.t. $A\omega(\theta^*) = -\lambda\omega(\theta^*)$;*
- (c) *Periodicity of $\varphi(A, \theta)$: $\varphi(A, \theta + \pi) = \varphi(A, \theta)$.*

Proof. From the definition in (3), we can confirm that, $A\omega(\theta)$ would have the same phase angle as $\omega(\theta + \varphi(A, \theta))$. It means that

$$A\omega(\theta) = \|A\omega(\theta)\|_2 \omega(\theta + \varphi(A, \theta)). \quad (5)$$

The pre-condition $\det(A) > 0$ ensures that $A\omega(\theta)$ is non-zero, then $\|A\omega(\theta)\|_2 > 0$ for all θ . Choosing $\lambda = \|A\omega(\theta^*)\|_2 > 0$ and considering $\omega(\theta + \pi) = -\omega(\theta)$, the sufficiency part of (a) and (b) can be easily proven, and their necessity parts would be obvious.

For (c), we can find the straight-forward derivation

$$\varphi(A, \theta + \pi) = \varphi(R^T(\pi)AR(\pi), \theta) = \varphi(A, \theta).$$

Thus the proof is completed. □

4. Phase-based Stability Analysis under Arbitrary Switching

Based on the concept of phase function, the main result of stability for system (1) under arbitrary switching will be introduced in this part. For system (1) to be stable under arbitrary switching, a necessary condition is that all the subsystems should be stable. Hence it is natural to propose the following assumption in this section.

Assumption 1: Matrices A_1, A_2, \dots, A_q are all Hurwitz matrices.

Under Assumption 1, all system matrices would satisfy $\det(A_i) > 0$ ($i \in \mathcal{Q}$). As a result, the properties in Lemmas 3.1 and 3.2 can be used in the stability analysis. Starting from the line-integral Lyapunov function in (2), we would firstly transform the Lyapunov function existence conditions (a), (b) and (c) in Section 2 into phase based criteria. The vector $p(x)$ considered in the following proposition is designed to be independent on the length $\|x\|_2$ of vector x , thus for $\theta = \text{atan}(x)$ we have $\varphi_p^*(\theta)|_{\theta=\text{atan}(x)} = \phi_p^*(\omega(\theta))|_{\theta=\text{atan}(x)} = \phi_p^*(x)$. Overall the Lyapunov function existence conditions can be described as the criteria of $\varphi_p^*(\theta)$ and $\varphi_\sigma(\theta)$ in Proposition 4.1.

Proposition 4.1: *If there exists $p(x)$ such that its phase function $\varphi_p^*(\theta)$ is continuous and satisfies*

$$\varphi_\sigma(\theta) - \frac{3\pi}{2} \leq \varphi_p^*(\theta) \leq \varphi_\sigma(\theta) - \frac{\pi}{2}, \quad (6)$$

$$-\frac{\pi}{2} < \varphi_p^*(\theta) < \frac{\pi}{2}, \quad (7)$$

for all $\theta \in [0, 2\pi)$, $\sigma(t) \in \mathcal{Q}$, and

$$\int_0^{2\pi} \tan \varphi_p^*(\theta) d\theta = 0, \quad (8)$$

then function (2) can be an appropriate line-integral Lyapunov function to ensure the stability of system (1). Moreover, system (1) is asymptotically stable if the inequalities in (6) are satisfied as strict ones.

Proof. See Appendix A. □

Remark 3: In the conditions of Proposition 4.1, the reason for using $\varphi_p^*(\theta)$ instead of $\varphi_p(\theta)$ is to simplify the expression in (6) and (7) by the special range property that $\varphi_p^*(\theta) \in [-\pi, \pi)$.

Remark 4: Based on Fig. 1 we can find that, condition (7) is provided to ensure that Lyapunov function $V(x)$ is positive definite, in other words, its level surface with lower energy is contained in level surface with higher energy. Condition (8) ensures that the level surface of $V(x)$ with the same energy is a closed circle. Condition (6) will guarantee that the system state x on a level surface moves inside that surface, in other words, $\dot{V}(x) \leq 0$.

In Proposition 4.1, the stability condition is described by the assumed phase function $\varphi_p^*(\theta)$ from the Lyapunov function. But in the actual case, what we can get are the phase functions of subsystems. To make the phase-based stability condition more applicable, we need to transform it into criteria about the phase function of subsystems $\varphi(A_i, \theta)$, ($i \in \mathcal{Q}$). Before proceeding to the main result, let us consider a necessary condition for the stability under arbitrary switching, which is related with the maximum and minimum values of all the subsystem phase functions. This condition can be viewed as a combination of (6) and (7) in Proposition 4.1.

Lemma 4.2: *A necessary condition for the stability of system (1) under arbitrary switching is*

$$\sup\{\varphi_{\max}(\theta) - \varphi_{\min}(\theta)\} \leq \pi, \quad (9)$$

where

$$\begin{aligned} \varphi_{\max}(\theta) &\triangleq \max\{\varphi(A_1, \theta), \varphi(A_2, \theta), \dots, \varphi(A_q, \theta)\}, \\ \varphi_{\min}(\theta) &\triangleq \min\{\varphi(A_1, \theta), \varphi(A_2, \theta), \dots, \varphi(A_q, \theta)\}. \end{aligned}$$

Proof. See Appendix B. □

The criterion expressed in (9) is in the form of phase function. To check the feasibility of this inequality we need to firstly get the values of all phase functions $\varphi(A_i, \theta)$ ($i = 1, 2, \dots, q$), for

$\theta \in [0, 2\pi)$, then find their maximum and minimum values $\varphi_{\max}(\theta)$ and $\varphi_{\min}(\theta)$. Finally the extreme value $\sup\{\varphi_{\max}(\theta) - \varphi_{\min}(\theta)\}$ can be obtained. Alternatively, we can also find the equivalent algebraic criterion if (9) is a strict inequality. The relation can be described in Lemma 4.3.

Lemma 4.3: *A necessary and sufficient condition for*

$$\sup\{\varphi_{\max}(\theta) - \varphi_{\min}(\theta)\} < \pi \quad (10)$$

to be satisfied is that $A_i A_j^{-1}$ has no negative real eigenvalue for all $1 \leq i < j \leq q$.

Proof. We choose an x such that $A_j x = \omega(\theta)$ can be satisfied. Hurwitz A_j is invertible, then the value of x can be obtained as $x = A_j^{-1} \omega(\theta)$. Furthermore, the difference of two phase functions $\varphi(A_i, \theta)$ and $\varphi(A_j, \theta)$ can be calculated as

$$\varphi(A_i, \theta) - \varphi(A_j, \theta) = \text{atan}(A_i A_j^{-1} \omega(\theta)) - \text{atan}(\omega(\theta)) = \varphi(A_i A_j^{-1}, \theta). \quad (11)$$

Inequality (10) means that $\varphi(A_i, \theta) - \varphi(A_j, \theta) \neq \pi$ for all $i, j \in \mathcal{Q}$, equivalently we have $\varphi(A_i A_j^{-1}, \theta) \neq \pi$ for all $i, j \in \mathcal{Q}$. By the relation (b) in Lemma 3.2, we know that matrix $A_i A_j^{-1}$ has no negative real eigenvalue for $i, j \in \mathcal{Q}$, or equivalently for all $1 \leq i < j \leq q$. The proof is thus completed. \square

The criterion in (9) can be viewed as a combination of (6) and (7). Our next step is to consider the condition represented by integral equation (8), and replace it with integral inequalities of $\cot \varphi_{\max}(\theta)$ and $\cot \varphi_{\min}(\theta)$. Based on the aforementioned phase functions $\varphi_{\max}(\theta)$ and $\varphi_{\min}(\theta)$, the main result of this paper can be stated as Theorem 4.4.

Theorem 4.4: *A necessary and sufficient condition for the stability of system (1) under arbitrary switching is that*

$$\sup\{\varphi_{\max}(\theta) - \varphi_{\min}(\theta)\} \leq \pi, \quad (12)$$

and

$$\inf\{\varphi_{\max}(\theta)\} \leq \pi \quad \text{or} \quad \int_0^{2\pi} \cot \varphi_{\max}(\theta) d\theta \geq 0, \quad \text{if } \inf\{\varphi_{\max}(\theta)\} > \pi, \quad (13)$$

$$\sup\{\varphi_{\min}(\theta)\} \geq \pi \quad \text{or} \quad \int_0^{2\pi} \cot \varphi_{\min}(\theta) d\theta \leq 0, \quad \text{if } \sup\{\varphi_{\min}(\theta)\} < \pi. \quad (14)$$

Moreover, system (1) is asymptotically stable if all the involved inequalities are satisfied as strict ones.

Proof. See Appendix C. \square

To check the stability condition in Theorem 4.4, firstly we need to get the expressions of $\varphi_{\max}(\theta)$ and $\varphi_{\min}(\theta)$ based on phase function $\varphi(A_i, \theta)$ of each subsystem. For criterion in (13), if the inequality $\inf\{\varphi_{\max}(\theta)\} \leq \pi$ holds, then there is no need to check $\int_0^{2\pi} \cot \varphi_{\max}(\theta) d\theta \geq 0$. Criterion

(13) does not hold if and only if

$$\int_0^{2\pi} \cot \varphi_{\max}(\theta) d\theta < 0 \quad \text{and} \quad \inf\{\varphi_{\max}(\theta)\} > \pi.$$

It is the same for criterion in (14).

Remark 5: The condition in (12) is provided to ensure that all the regional chattering dynamics are stable. Conditions in (13) and (14) can ensure that system state with spiralling dynamics does not diverge to infinity.

In Theorem 4.4, if $\inf\{\varphi_{\max}(\theta)\} \leq \pi$ then there is no need to check the inequality condition in (13). The same holds for $\inf\{\varphi_{\min}(\theta)\} \geq \pi$. Thus, as a special case of Theorem 4.4, we have the following simple sufficient stability condition.

Corollary 4.5: *System (1) is stable under arbitrary switching if the following inequalities are satisfied*

$$\inf\{\varphi_{\max}(\theta)\} \leq \pi, \tag{15}$$

$$\sup\{\varphi_{\max}(\theta) - \varphi_{\min}(\theta)\} \leq \pi, \tag{16}$$

$$\sup\{\varphi_{\min}(\theta)\} \geq \pi. \tag{17}$$

And system (1) is asymptotically stable if (16) is satisfied as strict inequality.

By two simple examples, now we discuss the application of phase-based condition in Proposition 4.1 and compare the results in Theorem 4.4 with several existing methods.

Example 1: To explain how the stability analysis conservativeness can be reduced by increasing the order of polynomial Lyapunov function, we will describe the phase functions of 2nd, 4th, 6th, 8th order polynomial Lyapunov functions (correspondingly in Chesi, Colaneri, Geromel, Middleton, & Shorten (2012) the value of parameter m should be set as $m = 1, 2, 3, 4$) based on an example with the following subsystems

$$A_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & -a \\ \frac{1}{a} & -1 \end{bmatrix}$$

where a is a parameter. The problem is to determine the maximum value of a^* for which the system is asymptotically stable for arbitrary switching signals. The results obtained by different order polynomial Lyapunov functions are shown in Table 1. The phase function of each polynomial Lyapunov

Table 1.: Maximum values of a^* obtained by different polynomial Lyapunov functions (Chesi et al., 2012)

m	1	2	3	4
a^*	5.8283	9.2911	9.6825	10.4105

function is plotted in Figure 3. We can find that, with the increase of a , the minimum value of $\varphi_{\min}(\theta)$ will move downward. From Proposition 4.1, it is obvious that, for an appropriate Lyapunov

function, its phase layout should be lower than $\varphi_{\min}(\theta) - \pi/2$ and satisfying $\int_0^{2\pi} \tan \varphi_p^*(\theta) d\theta = 0$. By increasing the value of m , the corresponding polynomial Lypapunov function will be more flexible. Consequently the gap between $\varphi_{\min}(\theta)$ and $\varphi_{\min}(\theta) - \pi/2$ can be smaller, and bigger value of a will be allowed. If we calculate the value of a by condition $\int_0^{2\pi} \cot \varphi_{\min}(\theta) d\theta = 0$, the critical value with 5 decimal places can be obtained as $a^* = 11.31149$.

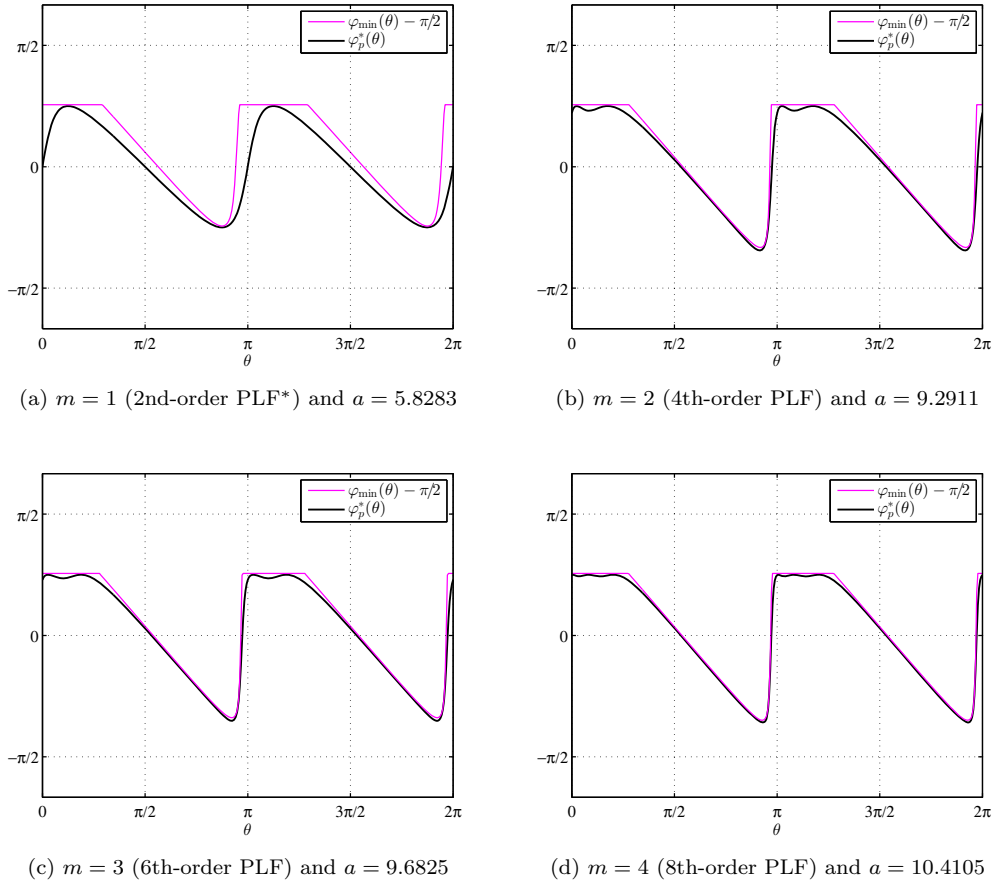


Figure 3.: Layouts of $\varphi_{\min}(\theta)$ and $\varphi_p^*(\theta)$ with different a value and m , (*PLF is the acronym of polynomial Lyapunov function)

Example 2: In this example, we will compare the phase-based stability condition in Theorem 4.4 with some existing methods in literature. Consider the following switched system model

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad A_{\sigma(t)} \in \{A_1, A_2\},$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -2-k & -1 \end{bmatrix}.$$

This example has been widely used in literature (e.g. Tanaka et al., 2009; Xie et al., 1997; Yfoulis & Shorten, 2004) to check the conservativeness of obtained results. Here we also choose it to exemplify

the result in Theorem 4.4 and make comparisons with existing methods.

(1). *Exemplification of the result in Theorem 4.4*

Based on the criteria in Proposition 4.1, the phase function $\varphi_p^*(\theta)$ of feasible Lyapunov function must lie within the grey region $[\varphi_{\max}(\theta) - \frac{3\pi}{2}, \frac{\pi}{2}]$ in Fig. 4, and satisfies the integral condition $\int_0^{2\pi} \tan \varphi_p^*(\theta) d\theta = 0$. The condition obtained from quadratic Lyapunov function (Tanaka et al., 2009) can guarantee the stability for $k \leq 3.82$, and its corresponding phase function is shown in Fig. 4. For the range $3.5 \leq k \leq 7.5$, we get $\sup\{\varphi_{\max}(\theta) - \varphi_{\min}(\theta)\} \in [1.08, 1.46]$, $\sup\{\varphi_{\min}(\theta)\} = 4.85$ and $\inf\{\varphi_{\min}(\theta)\} = 3.93$. Thus conditions (12) and (14) can be always satisfied, what we need to do is checking the integral condition in (13). By setting $\int_0^{2\pi} \cot \varphi_{\max}(\theta) d\theta = 0$, one can clearly find that the critical k value for (13) is $k^* = 6.98513$.

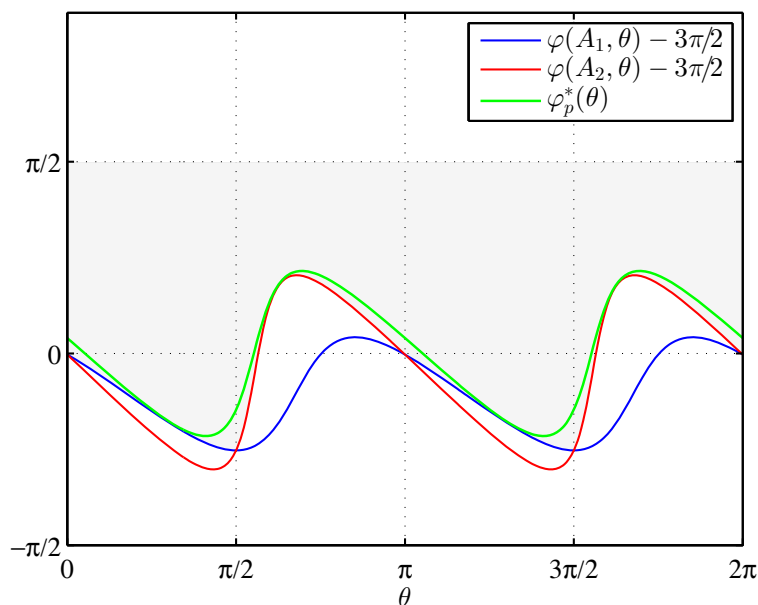


Figure 4.: Phase functions of the subsystems A_1 and A_2 and their common quadratic Lyapunov function $\varphi_p^*(\theta)$

(2). *Relation with Lyapunov function based methods*

Various novel Lyapunov functions have been proposed to improve the function flexibility. The piecewise Lyapunov function (Xie et al., 1997) constructed by double quadratic terms guarantees the stability for $k \leq 4.7$. Following this method, the result can be further improved to $k \leq 5.9$ if the nonlinear transformation in Zelentsovsky (1994) is combined.

The method proposed in Tanaka et al. (2009) and Chesi et al. (2012) is an extension of traditional quadratic Lyapunov function from second-order polynomial function to higher order ones. Theoretically, the conservativeness of stability condition in Tanaka et al. (2009) can be gradually reduced by increasing the order of adopted polynomial Lyapunov function. But when it is actually solved by the SOS Tools software, the improvement of analysis result will stop at some certain order. If we choose the function order to be even higher, as we can see in Fig. 5, the algorithm will crash and provide unreasonable results. This is mainly caused by the inevitable calculation-error

amplification of high order polynomials during the numerical iteration. The best result of this method is obtained by the tenth-order polynomial Lyapunov function, ensuring the stability for $k \leq 6.64$ which is below the value obtained by Theorem 4.4, as is shown in Fig. 5.

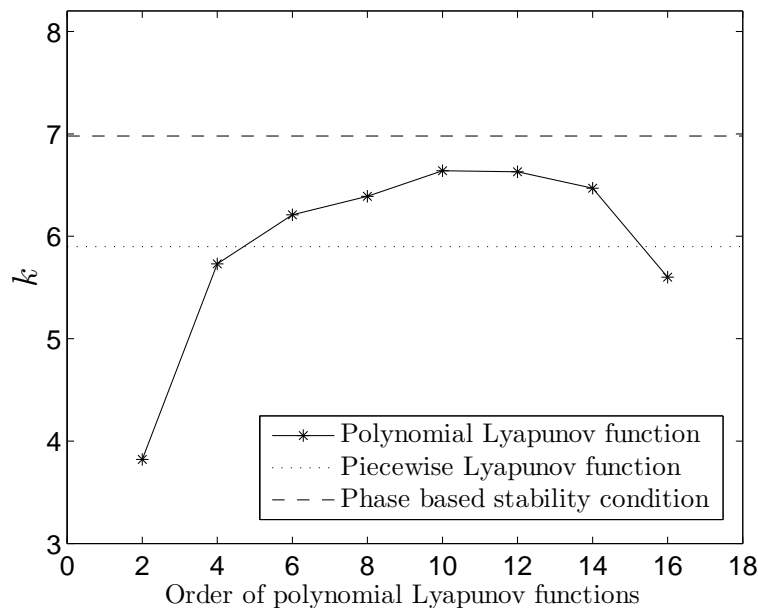


Figure 5.: Comparison among polynomial Lyapunov function (Tanaka et al., 2009), piecewise Lyapunov function (Xie et al., 1997), and phase-based condition in Theorem 4.4

(3). Relation with numerical methods

The numerical approach in Yfoulis & Shorten (2004) originates from the construction of a polyhedral Lyapunov function. Theoretically, the accuracy of k^* can be gradually improved by progressively choosing a larger number of *partition rays*. But this approach is computationally demanding in practical experiment. As we can see, one needs 40000 rays for a two-digit accuracy of $k^* = 6.98$ with time 1.13 seconds. But to achieve a three-digit accuracy of $k^* = 6.985$, the rays number 1500000 is required, and computational time will be longer than 43 seconds. The phase-based method in Theorem 4.4 can achieve results identical to that obtained by numerical method with infinite number of rays, and at the same time can get rid of the heavy burden of computation.

(4). Relation with polar coordinate based methods

Similar necessary and sufficient condition can be also found in Huang et al. (2010), which is obtained based on the polar coordinate model. But that condition is applicable to systems with only a pair of subsystems. Results in Yang et al. (2012) and Yang et al. (2014) can be viewed as the extension of Huang et al. (2010) to systems with finite number of subsystems. Compared with those results, the method studied here shows the advantage as a unified framework for the analysis of stability problem. So there is no need to concern about the specific types of eigenvalues that are defined in the above papers. And some assumptions for the subsystems can be also avoided, which means that a wider range of switched systems can be investigated.

5. Conclusions

Based on the unified framework of phase function, the problem of stability analysis for second-order switched system has been investigated. By considering and applying the inherent properties of phase function, necessary and sufficient condition for the stability of second-order switched systems under arbitrary switching has been obtained in a different approach. Compared with existing works, the stability condition obtained here shows advantages in terms of theoretical analysis and numerical computation.

Appendix A. Proof of Proposition 4.1

Proof. Based on the given phase function $\varphi_p^*(\theta)$ of $p(x)$, we can design the Lyapunov function as

$$V(x) = \|x\|_2 \exp(\rho(x)), \quad (\text{A1})$$

where $\rho(x) = \int_0^{\text{atan}(x)} \tan \varphi_p^*(\phi) d\phi$. Clearly, $\rho(x)$ is only related with $\text{atan}(x)$ and irrelevant to the value of $\|x\|_2$. Also, $\exp(\rho(x)) > 0$ for any x satisfying (7). Thus if we increase the value of $\|x\|_2$ along a constant angle $\theta = \text{atan}(x)$, $V(x)$ will increase proportionally. So the designed $V(x)$ in (A1) is radially unbounded. For $x = \|x\|_2 \omega(\text{atan}(x))$, we can find the equivalent expression

$$\begin{aligned} V(x) &= \|x\|_2 \exp(\rho(x)) \\ &= \|x\|_2 \cos \phi_p^*(x) \frac{\exp(\rho(x))}{\cos \phi_p^*(x)} \\ &= \|x\|_2 (\cos(\phi_p^*(x) + \text{atan}(x)) \cos(\text{atan}(x)) \\ &\quad + \sin(\phi_p^*(x) + \text{atan}(x)) \sin(\text{atan}(x))) \frac{\exp(\rho(x))}{\cos \phi_p^*(x)} \\ &= x^T \omega(\phi_p^*(x) + \text{atan}(x)) \frac{\exp(\rho(x))}{\cos \phi_p^*(x)} \\ &= p(x) \cdot x, \end{aligned} \quad (\text{A2})$$

where

$$p(x) \triangleq \omega(\phi_p^*(x) + \text{atan}(x)) \frac{\exp(\rho(x))}{\cos \phi_p^*(x)}. \quad (\text{A3})$$

Note that $\frac{\partial p_1(x)}{\partial x_2} = \frac{\partial p_2(x)}{\partial x_1}$, we can further obtain the line-integral expression of $V(x)$,

$$V(x) = \int_0^x p(x) dx. \quad (\text{A4})$$

By condition (7), we have $\cos \phi_p^*(x) > 0$, so $V(x) = \|p(x)\|_2 \|x\|_2 \cos \phi_p^*(x)$ is ensured to be positive-definite, which ensures condition (b) in Section 2. From (A3) we know that $p(x) = [p_1(x) \ p_2(x)]^T$

will not change with respect to $\|x\|_2$, thus

$$\begin{aligned}
 \frac{dp_1(x)}{d\theta} &= -\sin(\phi_p^*(x) + \theta) \frac{\exp(\rho(x))}{\cos \phi_p^*(x)} \left(\frac{d\phi_p^*(x)}{d\theta} + 1 \right) \\
 &\quad + \cos(\phi_p^*(x) + \theta) \frac{\exp(\rho(x))}{\cos \phi_p^*(x)} \tan \phi_p^*(x) \\
 &\quad + \cos(\phi_p^*(x) + \theta) \frac{\exp(\rho(x))}{\cos^2 \phi_p^*(x)} \sin \phi_p^*(x) \frac{d\phi_p^*(x)}{d\theta} \\
 &= -\sin(\phi_p^*(x) + \theta) \frac{\exp(\rho(x))}{\cos \phi_p^*(x)} \left(\frac{d\phi_p^*(x)}{d\theta} + 1 \right) \\
 &\quad + \cos(\phi_p^*(x) + \theta) \frac{\exp(\rho(x))}{\cos \phi_p^*(x)} \tan \phi_p^*(x) \left(\frac{d\phi_p^*(x)}{d\theta} + 1 \right) \\
 &= -\sin \theta \frac{\exp(\rho(x))}{\cos^2 \phi_p^*(x)} \left(\frac{d\phi_p^*(x)}{d\theta} + 1 \right),
 \end{aligned}$$

similarly

$$\frac{dp_2(x)}{d\theta} = \cos \theta \frac{\exp(\rho(x))}{\cos^2 \phi_p^*(x)} \left(\frac{d\phi_p^*(x)}{d\theta} + 1 \right).$$

Overall we have

$$\dot{p}(x) = \frac{\dot{\theta} \exp(\rho(x))}{\cos^2 \phi_p^*(x)} \left(\frac{d\phi_p^*(x)}{d\theta} + 1 \right) \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}. \quad (\text{A5})$$

Consequently, the time derivative of $V(x)$ can be obtained as

$$\dot{V}(x) = p(x) \cdot f_\sigma(x) + \dot{p}(x) \cdot x = p(x) \cdot f_\sigma(x). \quad (\text{A6})$$

From (A6) it can be found that $V(x)$ is continuously differentiable, then condition (a) in Section 2 is ensured. From (A1) we can know that the radial unbounded condition (c) is also satisfied. So $V(x)$ is an appropriate line-integral Lyapunov function candidate. From (6) and (A6), we know that $\dot{V}(x) = \|p(x)\|_2 \|f_\sigma(x)\|_2 \cos(\phi_p^*(x) - \phi_\sigma(x)) \leq 0$ for any x . Thus system (1) is ensured to be stable under arbitrary switching. Moreover, the strict form of (6) ensures $\dot{V}(x) < 0$, in which case the asymptotic stability of system (1) under arbitrary switching is guaranteed. The proof is then completed. \square

Appendix B. Proof of Lemma 4.2

In the proof of Lemma 4.2, we need the following result in (B1) to know the derivative of function $\text{atan}(A\omega(\theta))$ with respect to θ .

Lemma B.1: For any $A \in \mathbb{R}^{2 \times 2}$ with $\det(A) > 0$, the following equation holds

$$\frac{\partial \operatorname{atan}(A\omega(\theta))}{\partial \theta} = \frac{\det(A)}{\|A\omega(\theta)\|_2^2}. \quad (\text{B1})$$

Proof. Define a_{ij} ($i, j = 1, 2$) as the i -th row and j -th column element of A and $[a_1(\theta), a_2(\theta)]^T \triangleq A\omega(\theta)$. By calculation we get

$$\begin{aligned} \frac{\partial (a_2(\theta)/a_1(\theta))}{\partial \theta} &= \frac{(-a_{i21} \sin \theta + a_{i22} \cos \theta)a_1(\theta)}{a_1^2(\theta)} \\ &\quad - \frac{a_2(\theta)(-a_{i11} \sin \theta + a_{i12} \cos \theta)}{a_1^2(\theta)} \\ &= \frac{a_{11}a_{22} - a_{12}a_{21}}{a_1^2(\theta)} = \frac{\det(A)}{a_1^2(\theta)}. \end{aligned}$$

It follows that

$$\frac{\partial \operatorname{atan}(A\omega(\theta))}{\partial \theta} = \frac{a_1^2(\theta)}{a_1^2(\theta) + a_2^2(\theta)} \frac{\det(A)}{a_1^2(\theta)} = \frac{\det(A)}{\|A\omega(\theta)\|_2^2}.$$

Thus the proof is completed. \square

The **Proof of Lemma 4.2** will be summarized in the following part.

Proof. We construct the proof by a counter-example. In this example, we will find a chattering sequence $\sigma(t)$, along which, the system state $x(t)$ will diverge to infinity. The design process will be divided into three steps. In the first step we will find the sector region for the chattering; in the second step, we will prove that the chattering trajectory will magnify outward to the boundless direction. Finally in the third step, we will confirm the chatter signal $\sigma(t)$ to achieve such a chattering effect.

[Step 1. Find the region $\theta \in [\theta^*, \theta^* + \varepsilon_2]$ satisfying $\varphi_{\max}(\theta) - \varphi_{\min}(\theta) > \pi + \varepsilon_1]$

Note that both $\varphi_{\max}(\theta)$ and $\varphi_{\min}(\theta)$ are continuous functions. Thus the negation of (9) implies that there exist $\theta^* \in [0, 2\pi)$ and $\varepsilon_1 > 0$ satisfying

$$\varphi_{\max}(\theta^*) - \varphi_{\min}(\theta^*) = 2\varepsilon_1 + \pi. \quad (\text{B2})$$

Matrices A_i ($i \in \mathcal{Q}$) are all Hurwitz matrices, which ensure that, for any $\theta \in [0, 2\pi)$, $\|A_i\omega(\theta)\|_2^2 > 0$ and $\det(A_i) > 0$. Recalling the result in (B1), one gets the partial derivative expression of $\varphi(A_i, \theta)$,

$$\frac{\partial \varphi(A_i, \theta)}{\partial \theta} = -1 + \frac{\det(A_i)}{\|A_i\omega(\theta)\|_2^2}.$$

It means that

$$-1 < \frac{\partial \varphi(A_i, \theta)}{\partial \theta} \leq \eta - 1, \quad i \in \mathcal{Q}, \quad (\text{B3})$$

where $\eta \triangleq \max_{i \in \mathcal{Q}} \left\{ \frac{\det(A_i)}{\inf_{\theta} \{\|A_i \omega(\theta)\|_2^2\}} \right\}$. Further, we have

$$-\eta < \frac{\partial \varphi(A_i, \theta)}{\partial \theta} - \frac{\partial \varphi(A_j, \theta)}{\partial \theta} < \eta, \quad \forall \theta \in [0, 2\pi)$$

for all $i, j \in \mathcal{Q}$. Choose a real number $\varepsilon_2 = \frac{\varepsilon_1}{\eta} > 0$. Then considering (B2) and based on the mean value theorem, for any $\theta \in [\theta^*, \theta^* + \varepsilon_2]$, we have

$$\begin{aligned} \varphi_{\max}(\theta) - \varphi_{\min}(\theta) &= \max_{i, j \in \mathcal{Q}} \{\varphi(A_i, \theta) - \varphi(A_j, \theta)\} \\ &> \max_{i, j \in \mathcal{Q}} \{\varphi(A_i, \theta^*) - \varphi(A_j, \theta^*) - \eta(\theta - \theta^*)\} \\ &\geq \varphi_{\max}(\theta^*) - \varphi_{\min}(\theta^*) - \varepsilon_2 \eta \\ &= \pi + \varepsilon_1. \end{aligned} \tag{B4}$$

[Step 2. Prove that $\cot \varphi_{\min}(\theta) - \cot \varphi_{\max}(\theta) \geq \varepsilon_1$ for all $\theta \in [\theta^*, \theta^* + \varepsilon_2]$]

From part (a) in Lemma 3.2, one may find that, the Hurwitz property of A_i ($i \in \mathcal{Q}$) ensures $0 < \varphi(A_i, \theta) < 2\pi$, consequently $0 < \varphi_{\min}(\theta) \leq \varphi_{\max}(\theta) < 2\pi$. Combined with (B4), the above inequality leads to

$$0 < \varphi_{\min}(\theta) < \pi - \varepsilon_1, \quad \pi + \varepsilon_1 < \varphi_{\max}(\theta) < 2\pi.$$

Also note that $\cot \phi$ is a monotonically decreasing function for $\phi \in (0, \pi)$. Thus from (B4), it follows that

$$\cot \varphi_{\min}(\theta) > \cot(\varphi_{\max}(\theta) - \pi - \varepsilon_1) = \cot(\varphi_{\max}(\theta) - \varepsilon_1), \tag{B5}$$

for all $\theta \in [\theta^*, \theta^* + \varepsilon_2]$. Further applying mean value theorem, we know that, for any $\theta \in [\theta^*, \theta^* + \varepsilon_2]$ there exists $\tilde{\varphi} \in [\varphi_{\max}(\theta) - \varepsilon_1, \varphi_{\max}(\theta)]$ satisfying

$$\cot(\varphi_{\max}(\theta) - \varepsilon_1) = \cot \varphi_{\max}(\theta) + \varepsilon_1 \csc^2 \tilde{\varphi}. \tag{B6}$$

Note that $\csc^2 \tilde{\varphi} \geq 1$, thus combining (B5) and (B6) we get

$$\cot \varphi_{\min}(\theta) - \cot \varphi_{\max}(\theta) \geq \varepsilon_1. \tag{B7}$$

[Step 3. Design the switching signal $\sigma(t)$ such that $x(t)$ chatters to infinity in sector $[\theta^*, \theta^* + \varepsilon_2]$]

From Equation (4) in (Godbehere & Sastry (2010)) and using (5) for $A_{\sigma} \omega(\theta)$, we obtain

$$\dot{\theta} = \gamma^T(\theta) A_{\sigma} \omega(\theta) = \|A_{\sigma} \omega(\theta)\|_2 \sin \varphi_{\sigma}(\theta), \tag{B8}$$

$$\frac{1}{r} \dot{r} = \omega^T(\theta) A_{\sigma} \omega(\theta) = \|A_{\sigma} \omega(\theta)\|_2 \cos \varphi_{\sigma}(\theta), \tag{B9}$$

where $\gamma(\theta) \triangleq \begin{bmatrix} -\sin \theta & \cos \theta \end{bmatrix}^T$, $\dot{\theta} \triangleq \frac{d\theta}{dt}$ and $\dot{r} \triangleq \frac{dr}{dt}$, so that we conclude

$$\frac{1}{r}\dot{r} = \cot \varphi_{\sigma}(\theta)\dot{\theta}. \quad (\text{B10})$$

From (B8) we know that, if $\sigma(t)$ changes among \mathcal{Q} such that $\varphi_{\sigma}(\theta) = \varphi_{\max}(\theta) > \pi + \varepsilon_1$, then $\dot{\theta} < 0$. On the contrary, if $\varphi_{\sigma}(\theta) = \varphi_{\min}(\theta) < \pi - \varepsilon_1$ then $\dot{\theta} > 0$. Set the initial state as $x(0) = r(0)\omega(\theta^*)$, where $r(0) > 0$. And choose the switching signal in the following manner: if $\theta = \theta^*$ we choose the signal $\sigma(t)$ such that $\varphi_{\sigma}(\theta) = \varphi_{\min}(\theta)$ for all $\theta(x) \in [\theta^*, \theta^* + \varepsilon_2]$; if $\theta = \theta^* + \varepsilon_2$ we choose the signal $\sigma(t)$ such that $\varphi_{\sigma}(\theta) = \varphi_{\max}(\theta)$ for all $\theta(x) \in [\theta^*, \theta^* + \varepsilon_2]$. Denote the sequential switching times at $\theta = \theta^*$ as $t_0, t_2, t_4, \dots, t_{2m}, \dots$ and the sequential switching times at $\theta = \theta^* + \varepsilon_2$ as $t_1, t_3, t_5, \dots, t_{2m+1}, \dots$ where $m \in \mathbb{N}_0$, $t_0 = 0$ and $t_i < t_{i+1}$ for all $i \in \mathbb{N}_0$. Now, integrating (B10) from t_0 to t_{2m} and considering the relation in (B7), we then arrive at

$$\begin{aligned} \ln r(t_{2m}) - \ln r(t_0) &= \int_{t_0}^{t_{2m}} \cot \varphi_{\sigma}(\theta)\dot{\theta} dt \\ &= \sum_{i=0}^{m-1} \left(\int_{t_{2i}}^{t_{2i+1}} \cot \varphi_{\min}(\theta)\dot{\theta} dt + \int_{t_{2i+1}}^{t_{2i+2}} \cot \varphi_{\max}(\theta)\dot{\theta} dt \right) \\ &= \sum_{i=0}^{m-1} \left(\int_{\theta^*}^{\theta^* + \varepsilon_2} \cot \varphi_{\min}(\theta) d\theta + \int_{\theta^* + \varepsilon_2}^{\theta^*} \cot \varphi_{\max}(\theta) d\theta \right) \\ &= m \int_{\theta^*}^{\theta^* + \varepsilon_2} (\cot \varphi_{\min}(\theta) - \cot \varphi_{\max}(\theta)) d\theta \\ &> m\varepsilon_1\varepsilon_2. \end{aligned}$$

It means that $r(t_{2m}) > r(0) \exp(m\varepsilon_1\varepsilon_2)$. Apparently as t_{2m} goes to infinity, $r(t_{2m})$ increases to infinity. So, under the specially constructed switching law, system (1) is unstable. The necessity of (9) is then proven. \square

Appendix C. Proof of Theorem 4.4

Before the proof of Theorem 4.4, we need to explain why the *if* conditions in criteria (13) and (14) are needed. Clearly, if $\inf\{\varphi_{\max}(\theta)\} \leq \pi$, there may exist the case that $\varphi_{\max}(\theta) = \pi$ for some θ , then the function $\cot \varphi_{\max}(\theta)$ is not properly defined. It will be impossible for us to design the integral condition of $\cot \varphi_{\max}(\theta)$ similar to that in (8). In this case we can construct another modified function $\varphi_{\max\epsilon}(\theta)$ which is always bigger than π and satisfying $\pi < \varphi_{\max\epsilon}(\theta) < 2\pi$,

$$\varphi_{\max\epsilon}(\theta) \triangleq \max\{\varphi_{\max}(\theta), \pi + \epsilon\}, \quad (\text{C1})$$

where ϵ is a small positive value satisfying $0 < \epsilon < \pi$. Specially, if we choose ϵ as

$$\epsilon \triangleq \frac{1}{10} \min\left\{2\pi - \sup\{\varphi_{\max}(\theta)\}, \inf\{\varphi_{\min}(\theta)\}\right\}, \quad (\text{C2})$$

we can find that the integration of $\cot \varphi_{\max\epsilon}(\theta)$ from 0 to 2π will never be negative. This property can be summarized as Lemma C.1.

$$\int_0^{2\pi} \cot \varphi_{\max \epsilon}(\theta) \, \mathrm{d}\theta \geq 0, \quad (\text{C3})$$

Proof. To prove that $\int_0^{2\pi} \cot \varphi_{\max \epsilon}(\theta) d\theta \geq 0$, we need to construct an assistant periodical function $\varphi_{\max_1}(\theta) = \varphi_{\max_1}(\theta + \pi)$ which is not smaller than $\varphi_{\max \epsilon}(\theta)$ and satisfying

Assume that $\theta_\epsilon \in [0, \pi)$ is one intersection point of $\varphi_{\max}(\theta)$ and horizontal line $\pi + \epsilon$, then $\varphi_{\max_1}(\theta)$ can be constructed as a symmetric function around point $(\theta_\epsilon + \epsilon - \frac{\pi}{2}, \frac{3\pi}{2})$,

$$\varphi_{\max_1}(\theta) \triangleq \begin{cases} 2\pi - \epsilon, & \theta - \theta_\epsilon \in (\epsilon - \pi, 2\epsilon - \pi]; \\ \pi + \epsilon - (\theta - \theta_\epsilon), & \theta - \theta_\epsilon \in (2\epsilon - \pi, 0]; \\ \pi + \epsilon, & \theta - \theta_\epsilon \in (0, \epsilon]. \end{cases}$$

$$\varphi_{\max}(\theta) \leq \varphi_{\max}(\theta_{\epsilon}) - (\theta - \theta_{\epsilon}) = \varphi_{\max_1}(\theta). \quad (\text{C5})$$

Actually (C5) can be satisfied for any θ , thus

$$\int_0^{2\pi} \cot \varphi_{\max \epsilon}(\theta) d\theta \geq \int_0^{2\pi} \cot \varphi_{\max_1}(\theta) d\theta = 0. \quad (C6)$$

The proof is completed. \square

We now provide the **proof of Theorem 4.4** in the following part. For convenience, firstly we recall of content of it here.

Theorem 4.4: A necessary and sufficient condition for the stability of system (1) under arbitrary switching is that

$$\sup\{\varphi_{\max}(\theta) - \varphi_{\min}(\theta)\} \leq \pi, \quad (12)$$

and

$$\inf\{\varphi_{\max}(\theta)\} \leq \pi \quad \text{or} \quad \int_0^{2\pi} \cot \varphi_{\max}(\theta) d\theta \geq 0, \quad \text{if } \inf\{\varphi_{\max}(\theta)\} > \pi, \quad (13)$$

$$\sup\{\varphi_{\min}(\theta)\} \geq \pi \quad \text{or} \quad \int_0^{2\pi} \cot \varphi_{\min}(\theta) d\theta \leq 0, \quad \text{if } \sup\{\varphi_{\min}(\theta)\} < \pi. \quad (14)$$

Moreover, system (1) is asymptotically stable if all the involved inequalities are satisfied as strict ones.

Proof. (Sufficiency)

The proof of sufficiency will be divided into three steps. In the first step, we will construct new phase functions $\hat{\varphi}_{\max}(\theta)$ and $\hat{\varphi}_{\min}(\theta)$ which can avoid the *if* conditions in criteria (13) and (14). In the second step, we will find the appropriate phase function $\varphi_p^*(\theta)$ of Lyapunov function which is located between $\hat{\varphi}_{\max}(\theta)$ and $\hat{\varphi}_{\min}(\theta)$, and criterion $\int_0^{2\pi} \tan \varphi_p^*(\theta) d\theta = 0$ in Proposition 4.1. In the third step, the case of asymptotic stability will be discussed.

[**Step 1.** Construct new functions $\hat{\varphi}_{\max}(\theta)$ and $\hat{\varphi}_{\min}(\theta)$ such that $\cot \hat{\varphi}_{\min}(\theta)$ and $\cot \hat{\varphi}_{\max}(\theta)$ are well defined and satisfying $\int_0^{2\pi} \cot \hat{\varphi}_{\min}(\theta) d\theta \leq 0 \leq \int_0^{2\pi} \cot \hat{\varphi}_{\max}(\theta) d\theta$].

Define $\varphi_{\min \epsilon}(\theta)$ as

$$\varphi_{\min \epsilon}(\theta) \triangleq \min \{ \varphi_{\min}(\theta), \pi - \epsilon \},$$

where ϵ is defined in (C2). Following the proof of Lemma C.1, we know that, if $\sup\{\varphi_{\min}(\theta)\} \geq \pi$, then

$$\int_0^{2\pi} \cot \varphi_{\min \epsilon}(\theta) d\theta \leq 0. \quad (C7)$$

Define the following new functions to cover both cases of the *if* conditions in (13) and (14)

$$\hat{\varphi}_{\max}(\theta) \triangleq \begin{cases} \varphi_{\max \epsilon}(\theta), & \inf\{\varphi_{\max}(\theta)\} \leq \pi; \\ \varphi_{\max}(\theta), & \inf\{\varphi_{\max}(\theta)\} > \pi, \end{cases} \quad (\text{C8})$$

$$\hat{\varphi}_{\min}(\theta) \triangleq \begin{cases} \varphi_{\min \epsilon}(\theta), & \sup\{\varphi_{\min}(\theta)\} \geq \pi; \\ \varphi_{\min}(\theta), & \sup\{\varphi_{\min}(\theta)\} < \pi. \end{cases} \quad (\text{C9})$$

By (C3) and (13) we can assert that

$$\int_0^{2\pi} \cot \hat{\varphi}_{\max}(\theta) d\theta \geq 0. \quad (\text{C10})$$

Similarly by (14) and (C7), for any $\varphi_{\min}(\theta)$, one has

$$\int_0^{2\pi} \cot \hat{\varphi}_{\min}(\theta) d\theta \leq 0. \quad (\text{C11})$$

Moreover, the definitions in (C8) and (C9) also ensure that

$$0 < \hat{\varphi}_{\min}(\theta) < \pi < \hat{\varphi}_{\max}(\theta) < 2\pi, \quad (\text{C12})$$

and

$$\hat{\varphi}_{\min}(\theta) < \varphi_{\sigma}(\theta) < \hat{\varphi}_{\max}(\theta). \quad (\text{C13})$$

[**Step 2.** Construct the desired phase function $\varphi_p^*(\theta)$ based on the weighted sum of $\hat{\varphi}_{\max}(\theta)$ and $\hat{\varphi}_{\min}(\theta)$]

Construct the new variable $\hat{\varphi}(\theta, \alpha)$ which is a continuous function of θ and $\alpha \in [0, 1]$,

$$\hat{\varphi}(\theta, \alpha) \triangleq \alpha \cdot (\hat{\varphi}_{\max}(\theta) - \frac{3\pi}{2}) + (1 - \alpha) \cdot (\hat{\varphi}_{\min}(\theta) - \frac{\pi}{2}). \quad (\text{C14})$$

Based on (C10) and (C11), we know $\hat{\varphi}(\theta, \alpha)$ possesses the property that

$$\int_0^{2\pi} \tan \hat{\varphi}(\theta, 0) d\theta \geq 0, \quad \int_0^{2\pi} \tan \hat{\varphi}(\theta, 1) d\theta \leq 0.$$

The function $\int_0^{2\pi} \tan \hat{\varphi}(\theta, \alpha) d\theta$ is continuous with respect to $\alpha \in [0, 1]$. Based on the intermediate value theorem, we know that there exists an $\tilde{\alpha} \in [0, 1]$ satisfying

$$\int_0^{2\pi} \tan \hat{\varphi}(\theta, \tilde{\alpha}) d\theta = 0. \quad (\text{C15})$$

Thus criterion (8) is satisfied. Recalling the definition in (C2), we may find the relation

$$\varphi_{\min}(\theta) \geq 10\epsilon, \quad \varphi_{\max}(\theta) \leq 2\pi - 10\epsilon.$$

Then the following inequalities will also be ensured,

$$\begin{aligned}\varphi_{\max}(\theta) - (\pi - \epsilon) &\leq \pi - 9\epsilon \leq \pi, \\ (\pi + \epsilon) - \varphi_{\min}(\theta) &\leq \pi - 9\epsilon \leq \pi, \\ (\pi + \epsilon) - (\pi - \epsilon) &\leq 2\epsilon \leq \pi.\end{aligned}$$

Together with the condition in (12), we know that $\hat{\varphi}_{\max}(\theta) - \hat{\varphi}_{\min}(\theta) \leq \pi$, which equivalently means

$$\hat{\varphi}_{\max}(\theta) - \frac{3\pi}{2} \leq \hat{\varphi}_{\min}(\theta) - \frac{\pi}{2}. \quad (\text{C16})$$

The definition in (C14) ensures that $\hat{\varphi}(\theta, \tilde{\alpha})$ has value between $\hat{\varphi}_{\max}(\theta) - \frac{3\pi}{2}$ and $\hat{\varphi}_{\min}(\theta) - \frac{\pi}{2}$,

$$\hat{\varphi}_{\max}(\theta) - \frac{3\pi}{2} \leq \hat{\varphi}(\theta, \tilde{\alpha}) \leq \hat{\varphi}_{\min}(\theta) - \frac{\pi}{2}. \quad (\text{C17})$$

So together with (C13), inequality (C17) indicates that criterion (6) can be satisfied,

$$\varphi_{\sigma}(\theta) - \frac{3\pi}{2} \leq \hat{\varphi}(\theta, \tilde{\alpha}) \leq \varphi_{\sigma}(\theta) - \frac{\pi}{2}. \quad (\text{C18})$$

Considering (C12) and (C17), we know that criterion (7) is also ensured,

$$-\frac{\pi}{2} < \hat{\varphi}(\theta, \tilde{\alpha}) < \frac{\pi}{2}. \quad (\text{C19})$$

Combined with criterion (C15), the criteria (C18) and (C19) imply that $\hat{\varphi}(\theta, \tilde{\alpha})$ can be regarded as the desired phase function $\varphi_p^*(\theta)$ which guarantees the stability of system (1). Based on the statement in Proposition 4.1, the sufficiency of (12)–(14) for stability is thus proven.

[Step 3. Proof of asymptotic stability in the case of strict inequalities]

Moreover, if all the involved inequalities are satisfied as strict ones, parameter $\tilde{\alpha}$ in (C15) will be restricted by $0 < \tilde{\alpha} < 1$ and the constraint in (C16) will be strengthened as strict inequality. Therefore, (C17) becomes strict inequality, and so is (C18). By Proposition 4.1, system (1) can be guaranteed to be asymptotically stable if all the involved inequalities are satisfied as strict ones.

(Necessity)

Lemma 4.2 ensures the necessity of (12). The remaining work is the proof of the necessity of (13) and (14). We finish this part by a pseudo-proposition of (13) which can be stated as

$$\inf\{\varphi_{\max}(\theta)\} > \pi \quad \text{and} \quad \int_0^{2\pi} \cot \varphi_{\max}(\theta) d\theta > 0. \quad (\text{C20})$$

From (B8), if the switching sequence $\sigma(t)$ is chosen to ensure $\varphi_{\sigma}(\theta) = \varphi_{\max}(\theta)$, then $\dot{\theta} < 0$ can be guaranteed by the first inequality in (C20). Denote the initial state as $x(0) = r(0)\omega(\theta_0)$ where $r(0) > 0$, and the time sequence at $\theta = \theta_0$ as $t_0, t_1, t_2, \dots, t_m, \dots$ where $m \in \mathbb{N}_0$, $t_0 = 0$ and $t_i < t_{i+1}$ for all $i \in \mathbb{N}_0$. Based on the polar coordinate model, we get $r(t_m) = r(0) \exp(m\varepsilon)$, where

$\varepsilon = \int_0^{2\pi} \cot \varphi_{\max}(\theta) d\theta > 0$. It means that $r(t_m)$ goes to infinity as t_m increases and then system (1) under the designed switching strategy is unstable. Similarly, system (1) without condition (14) is unstable under the switching sequence $\sigma(t)$ that satisfies $\varphi_\sigma(\theta) = \varphi_{\min}(\theta)$. The proof of necessity is then completed. \square

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